Pergamon

Int. J. Solids Structures Vol. 31, No. 21, pp. 2945–2950, 1994

0020–7683/94 $7.00 + .00

0020–7683(94)00069–7

SHOULD LOAD REMAIN CONSTANT WHEN A THIN-WALLED OPEN-PROFILE COLUMN BUCKLES?

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(*Received* 30 *December* 1993; *in revised form* 14 *April* 1994)

Abstract—This paper addresses the fundamental question of whether the applied load should necessarily be assumed to remain constant for bifurcation buckling analysis of thin-walled open-profile columns via the work–energy approach. The constant load condition has been recently used as a fundamental requirement for a valid analysis by Goto and Chen (1989, *Int. J. Solids Structures* **25,** 621–634) to dispute a theory proposed by Ojalvo (1989, *ASME J. Appl. Mech.* **56,** 633–638). This paper establishes that neither constant load nor constant distance need be insisted upon for a valid analysis. It is permissible to assume either constant load or constant distance (or any other compatible configuration with second order changes in loads and distance) for a valid analysis. It is essential to ensure that the appropriate strain energy and work expressions are adopted.

1. INTRODUCTION

Goto and Chen (1989) have attempted to counter Ojalvo's (1989) dismissal of the Wagner effect in the buckling theory of thin-walled open-profile columns, by arguing that the axial load should not change even by a second-order magnitude as the column buckles if the analysis via the work–energy approach is to be valid. In fact, constraining the load to a constant for bifurcation buckling analysis is a widely adopted practice [see, for example, Timoshenko and Gere (1961); Thompson and Hunt (1973); Chajes (1974); Washizu (1982); Trahair and Bradford (1988)]. Alwis and Wang (1995), however, have recently pointed out that there is no need to insist on either constancy of load or constancy of distance between the bar ends when considering the deformed configuration for bifurcation flexural buckling analysis of axially loaded bars via energy approach. As long as the buckling mode satisfies the equilibrium, compatibility and constitutive relations, the correct solution can be obtained.

The present paper addresses the question of whether the above conclusion on second-order load changes during buckling still applies for buckling of thin-walled open-profile columns. This fundamental question is important, in view of the aforementioned assertion used by Goto and Chen (1989) with reference to the same phenomenon, to study the validity of a buckling theory. Note that the extension of the earlier study on flexural buckling of bars to buckling of thin-walled open-profile columns is not trivial since the latter type of columns may buckle in flexural, torsional or combined modes. Twisting causes the cross-section to warp, thus, even when no shear distortion in the middle planes is assumed, points on an original plane section would no longer be contained in a plane during torsional deformation, unlike in pure flexural deformation.

2. STRAIN–DISPLACEMENT RELATIONSHIP

Consider a straight, thin-walled open-profile member as shown in Fig. 1. Adopting the Cartesian coordinate system (x, y, z) defined for the initial configuration of the member with z-axis along the centroidal axis, the incremental displacement components (u, v, w) in the (x, y, z) directions of a point can be expressed as (Goto and Chen, 1989),

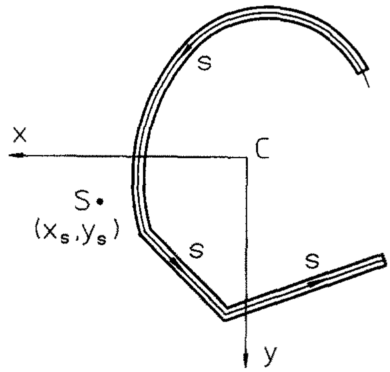

Fig. 1. Cross-section of a thin-walled open-profile member.

$$u = u_s - (y - y_s)\theta = u_c - y\theta \tag{1a}$$

$$v = v_s + (x - x_s)\theta = v_c + x\theta \tag{1b}$$

$$w = w_c - xu_s' - yv_s' - \omega\theta', \tag{1c}$$

where θ is the incremental rotational angle about the z-axis and $\omega(s)$ is a warping function, subscripts s and c denote quantities on the shear centre and centroid, respectively, and $(\cdot)'$ denotes differentiation with respect to z. The term w_c is commonly taken as the centroidal deflection in the z-direction. It should be realized that w_c is an imagined displacement representing the average of axial displacements at all material points of the cross-section, irrespective of whether the centroidal axis passes through the material or not.

For convenience of mathematical manipulations, the axes (x, y) are chosen to coincide with the principal axes of the cross-section. As a result, denoting integration over the cross-sectional area A by $\int_A(\cdot)\,\mathrm{d}A$,

$$\int_A x\,\mathrm{d}A = 0, \quad \int_A y\,\mathrm{d}A = 0, \quad \int_A xy\,\mathrm{d}A = 0. \tag{2a–c}$$

The origin of the profile coordinate s can be selected such that the warping function ω, satisfies

$$\int_A \omega\,\mathrm{d}A = 0. \tag{3}$$

The point S, being the shear center, satisfies the conditions

$$\int_A \omega x\,\mathrm{d}A = 0, \quad \int_A \omega y\,\mathrm{d}A = 0. \tag{4a,b}$$

Biot's incremental strain definition is employed and the strain–displacement relation applicable to fibres, which are initially parallel to the member axis, are derived below. All displacement and strain variables refer to the coordinate locations of the straight member of length L under the axial load P. Let E be the incremental elastic modulus associated with Biot's incremental strain.

Neglecting the terms of order higher than $\partial w/\partial z$, $(\partial u/\partial z)^2$ and $(\partial v/\partial z)^2$, the longitudinal normal strain increment can be expressed in the form

$$\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial z}\right)^2. \tag{5}$$

This is a well-known equation and is the same as that implemented by Goto and Chen (1989). Substitution of eqns (1) in eqn (5) leads to

$$\varepsilon_z = -xu_s'' - yv_s'' - \omega\theta'' + \mathrm{O}_1^2(z) + \mathrm{O}_2^2(x, y, z), \tag{6}$$

where

$$\mathrm{O}_1^2(z) = w_c' + \tfrac{1}{2}(u_c'^2 + v_c'^2) \tag{7a}$$

$$\mathrm{O}_2^2(x, y, z) = \tfrac{1}{2}(x^2 + y^2)\theta'^2 + (xv_c' - yu_c')\theta'. \tag{7b}$$

Note that $\mathrm{O}_1^2(z)$ is defined in terms of centroidal parameters whereas $\mathrm{O}_2^2(x, y, z)$ vanishes at the centroid.

3. RELATIONSHIP BETWEEN CHANGES IN LOAD AND SUPPORT DISTANCE

It follows from eqn (7a) that the change in distance between the ends of the centroidal line is given by

$$\Delta w_c = \int_0^L w_c' \, \mathrm{d}z = \int_0^L \mathrm{O}_1^2(z) \, \mathrm{d}z - \frac{1}{2}\int_0^L (u_c'^2 + v_c'^2) \, \mathrm{d}z. \tag{8}$$

Note that the end points of the centroidal line are the centroids of the end sectional surfaces, in accordance with the definition of w_c as the average displacement.

The second-order change in the load P during deformation is given by

$$\Delta P = \int_A E\varepsilon_z \, \mathrm{d}A. \tag{9}$$

Substitution of eqn (6) into eqn (9) yields

$$\Delta P = \mathrm{O}_1^2(z)EA + \tfrac{1}{2}r^2\theta'^2 EA, \tag{10}$$

where r is the radius of gyration of the section about the centroidal axis. Note that the last term on the right-hand side of eqn (10) arose from the integration of $\mathrm{O}_2^2(x, y, z)$ over the sectional area, i.e.

$$\int_A \mathrm{O}_2^2(x, y, z) \, \mathrm{d}A = \tfrac{1}{2}Ar^2\theta'^2. \tag{11}$$

Substitution of $\mathrm{O}_1^2(z)$ from eqn (10) into eqn (8) furnishes

$$\Delta w_c - \frac{L}{EA}\Delta P = -\frac{1}{2}\int_0^L (u_c'^2 + v_c'^2 + r^2\theta'^2) \, \mathrm{d}z = \delta. \tag{12}$$

Equation (12) gives the relationship between the changes in centroidal end distance and the load, as the column deforms.

4. BUCKLING ANALYSIS

The incremental work done by the load can be expressed as

$$W = -P\Delta w_c, \tag{13}$$

where P is the compressive axial load. Note that the second-order change in P does not appear in eqn (13) since its product with the second-order displacement increments would produce a fourth-order term. By substituting from eqn (8), the work done in eqn (13) can be expressed as

$$W = -P\int_0^L O_1^2(z)\,dz + \frac{P}{2}\int_0^L (u_c'^2 + v_c'^2)\,dz. \tag{14}$$

In view of eqns (8) and (10), the second-order term $O_1^2(z)$ here can be interpreted as a term that represents second-order changes in centroidal end distance or load.

The corresponding increment of strain energy of the buckled bar is given by

$$U = \int_V \left(-\frac{P}{A} + \frac{1}{2}E\varepsilon_z\right)\varepsilon_z\,dV + \frac{1}{2}\int_0^L GJ\theta'^2\,dz, \tag{15}$$

where V denotes the volume of the bar and GJ is the torsional rigidity. By substituting eqn (6) into eqn (15) and neglecting terms of order higher than second, the strain energy becomes

$$U = -P\int_0^L O_1^2(z)\,dz - \frac{P}{A}\int_V O_2^2(x,y,z)\,dV + \frac{1}{2}\int_0^L (EI_y u_s''^2 + EI_x v_s''^2 + EI_\omega \theta''^2 + GJ\theta'^2)\,dz, \tag{16}$$

in which I_x and I_y are the second moments of area about the x and y axes, respectively, and I_ω is the warping section constant.

In view of eqns (14) and (16), the total incremental energy functional is given by

$$\Pi = U - W \tag{17}$$

$$= -\frac{P}{A}\int_V O_2^2(x,y,z)\,dV + \frac{1}{2}\int_0^L [EI_y u_s''^2 + EI_x v_s''^2 + EI_\omega \theta''^2 + GJ\theta'^2 - P(u_c'^2 + v_c'^2)]\,dz.$$

It is worthwhile noting that a crucial difference between purely flexural buckling and flexural–torsional buckling is that, in the former, the $O_2^2(x,y,z)$ term does not arise in the formulation. Recall that eqn (11) produces

$$\frac{P}{A}\int_V O_2^2(x,y,z)\,dV = \tfrac{1}{2}Pr^2\int_0^L \theta'^2\,dz. \tag{18}$$

By substituting eqn (18) into eqn (17), the energy functional can be expressed as

$$\Pi = \frac{1}{2}\int_0^L [EI_y u_s''^2 + EI_x v_s''^2 + EI_\omega \theta''^2 + GJ\theta'^2 - P(u_c'^2 + v_c'^2 + r^2\theta'^2)]\,dz. \tag{19}$$

This energy expression is essentially the same as that obtained by Goto and Chen (1989) and others. The last term on the right-hand side of eqn (19) is the well-known Wagner term. Readers who are familiar with Ojalvo's arguments against the Wagner hypothesis should note that the term in eqn (18) is exactly the difference between the Wagner term and the alternative term proposed by Ojalvo (1989).

By minimising the energy functional with respect to the generalized displacement components and ensuring that the kinematic boundary conditions are satisfied, the buckling solution can be obtained in the usual manner.

It follows from the foregoing that any second-order change Δw_c and ΔP may be admitted provided that the relationship given in eqn (12) is satisfied. Indeed, neither constant load nor constant distance need be insisted upon for a valid analysis.

5. IMPLICATIONS OF CONSTRAINING SECOND-ORDER CHANGES

5.1. *Implications of the constant load condition*

Consider the assumption that the load remains constant during buckling (i.e. $\Delta P = 0$). As dictated by eqn (12), the change in distance must be

$$\Delta w_c = -\frac{1}{2}\int_0^L (u_c'^2 + v_c'^2 + r^2\theta'^2)\,dz = \delta. \tag{20}$$

Furthermore, eqn (10) leads to

$$O_1^2(z) = -\tfrac{1}{2}r^2\theta'^2. \tag{21}$$

In view of eqn (20), the incremental work done of eqn (13) takes the following form:

$$W = -P\Delta w_c = \frac{P}{2}\int_0^L (u_c'^2 + v_c'^2 + r^2\theta'^2)\,dz. \tag{22}$$

On substitution of eqns (18) and (21) into eqn (16), the strain energy functional takes the familiar form

$$U = \frac{1}{2}\int_0^L (EI_y u_s''^2 + EI_x v_s''^2 + EI_\omega \theta''^2 + GJ\theta'^2)\,dz. \tag{23}$$

This shows that the conventional form of strain energy functional (Bleich, 1952) implicitly presumes constant load during buckling.

It is observed that both derivations of Ojalvo (1989) and Goto and Chen (1989) have used the conventional strain energy functional and by so doing, they have committed themselves to the assumption of constant load during buckling.

5.2. *Implications of constant centroidal length condition*

On the other hand, if one presumes that there is no change in distance between the centroidal ends during buckling (i.e. $\Delta w_c = 0$), then eqn (12) shows that the change in load must be given by

$$\Delta P = -\frac{EA}{L}\delta. \tag{24}$$

Furthermore, eqn (8) leads to

$$\int_0^L O_1^2(z)\,dz = \frac{1}{2}\int_0^L (u_c'^2 + v_c'^2)\,dz. \tag{25}$$

For this case, by virtue of null Δw_c, the incremental work done is

$$W = 0. \tag{26}$$

In view of eqns (18) and (25), the incremental strain energy functional in eqn (16) becomes

$$U = \frac{1}{2}\int_0^L [EI_y u_s''^2 + EI_x v_s''^2 + EI_\omega \theta''^2 + GJ\theta'^2 - P(u_c'^2 + v_c'^2 + r^2\theta'^2)]\,dz. \tag{27}$$

Note that the strain energy functional in eqn (27) is the same as the total potential energy functional in the general case.

6. CONCLUDING REMARKS

Hitherto, the applied axial load has been invariably assumed as a constant for bifurcation buckling analysis of columns via work–energy approach. Recently Alwis and Wang (1995) have shown that there is no need to insist on either constancy of load or constancy of distance between the bar ends for bifurcation flexural buckling analysis via work–energy approach. The present study establishes that this conclusion is also valid for flexural–torsional buckling of thin-walled open-profile columns.

It is permissible to assume either constant load or constant distance (or any other compatible configuration with second-order changes in load and distance) for a valid analysis. It is essential to ensure that the appropriate strain energy and work expressions are adopted, as pointed out in Section 5.

With reference to Goto and Chen's (1989) arguments against Ojalvo's (1989) derivation, the foregoing conclusion establishes that allowing for a second-order change in load is not an error in itself. The writers would like to emphasize that the Wagner term appears even when a second-order change in load is allowed. The actual error made by Ojalvo, together with other issues brought up by him such as nonavailability of a free-body diagram to show the Wagner effect, will be addressed in a future paper.

It is well known that the buckling load is a function of the member length. In the classical buckling analysis, the column length is taken as the original undeformed length. In the present analysis, the column length L at the point of incipient buckling was not specifically identified as an undeformed length or a deformed length, yet the energy functional derived is the same as that obtained in the classical analysis. As such, if one wishes to include the effect of prebuckling axial shortening of the column, one needs only to use the shortened column length and the appropriate incremental elastic moduli (Bažant 1970) in the classical solutions.

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